

# A Tiling Proof of Binomial Identities related to the Lucas cube

Andrzej P. Kisielewicz

Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski  
ul. Z. Szafrana 4a, 65-516 Zielona Góra, Poland  
A.Kisielewicz@wmie.uz.zgora.pl

## Abstract

Using a cube tiling of  $\mathbb{R}^n$  constructed by Lagarias and Shor a tiling proof of three well-known binomial identities related to the Lucas cube is given.

*Key words:* cube tiling, binomial identity, Lucas cube.

## 1 Introduction

A cube tiling of  $\mathbb{R}^n$  is a family of cubes  $[0, 2)^n + T = \{[0, 2)^n + t : t \in T\}$ , where  $T \subset \mathbb{R}^n$ , which fill in the whole space without gaps and overlaps. In [4] Lagarias and Shor constructed a cube tiling code of  $\mathbb{R}^n$ . In this note we show how it can be used to prove the following well-known identities:

$$\sum_{k \geq 0} \binom{n-k}{k} \frac{n}{n-k} 2^k = 2^n + (-1)^n, \quad (1.1)$$

$$\sum_{k \geq 0} \binom{n-k}{k} 2^k = 1/3(2^{n+1} + (-1)^n) \quad (1.2)$$

and

$$\sum_{k \geq 0} \binom{n-k}{k} \frac{k}{n-k} 2^k = 1/3(2^n + (-1)^n 2). \quad (1.3)$$

The code of Lagarias and Shor is constructed as follows. Let  $n \geq 3$  be an odd positive integer, and let  $A$  be an  $n \times n$  circulant matrix of the form

$A = A(n) = \text{circ}(1, 2, 0, \dots, 0)$ . Let  $A^T$  be the transpose of  $A$ . By  $V(A)$  and  $V(A^T)$  we denote the sets of all the sums of the different rows in  $A$  and  $A^T$ , respectively. Moreover, we add to these sets the vector  $(0, \dots, 0)$ . Let

$$V = V_e(A) \cup (V_o(A^T) + (2, \dots, 2)) \pmod{4},$$

where  $V_e(A)$  denotes the set of all vectors in  $V(A)$  with an even number of 3s, and  $V_o(A^T)$  is the set of all vectors in  $V(A^T)$  with an odd number of 0s. We will refer to the code  $V$  as the *Lagarias-Shor cube tiling code*. This code has very interesting applications. Originally in [4] it was used to design a certain cube tiling of  $\mathbb{R}^n$  that was the basis for estimating distances between cubes in cube tilings of  $\mathbb{R}^n$ . Recently in [3] the Lagarias-Shor cube tiling code was used to construct interesting partitions and matchings of an  $n$ -dimensional cube.

To obtain a cube tiling of  $\mathbb{R}^n$  from the code  $V$ , let  $T = V - \mathbf{1} + 4\mathbb{Z}^n$ , where  $\mathbf{1} = (1, \dots, 1)$ . It follows from [4, Proposition 3.1 and Theorem 4.1] that  $[0, 2)^n + T$  is a cube tiling of  $\mathbb{R}^n$ . (To be precisely, a tiling considered in [4] is of the form  $[0, 1)^n + T'$ , where  $T' = \frac{1}{2}V + 2\mathbb{Z}^n$ , but  $[0, 2)^n + T = [0, 2)^n + 2T' - \mathbf{1}$ .)

All three identities (1.1)–(1.3) are related to the *Lucas cube*  $\Lambda_n$ . This is a graph whose vertices are all elements of  $\{0, 1\}^n$  which do not contain two consecutive 1s as well as vertices having 1 at the first and last position simultaneously. It is known that  $\binom{n-k}{k} \frac{n}{n-k}$  is the number of all vertices  $v$  in the Lucas cube  $\Lambda_n$  of weight  $k$ , i.e., containing  $k$  1s. This is also the number of all  $k$ -element subsets of the set  $[n] = \{1, \dots, n\}$  without two consecutive integers and which do not contain the pair  $1, n$  ([5]). The number of all vertices in  $\Lambda_n$  of weight  $k$  which have 1 at the  $i$ th position,  $i \in [n]$ , is equal to  $\binom{n-k}{k} \frac{k}{n-k}$ , while  $\binom{n-k}{k}$  is the number of all vertices in  $\Lambda_n$  of weight  $k$  which have 0 at the  $i$ th position.

## 2 Tiling proofs

Since the Lagarias-Shor cube tiling code is defined for odd numbers, we prove identities (1.1)–(1.3) for odd and even positive integers separately.

*Proof of (1.1) for  $n \geq 3$  odd.* We intersect the cube  $[0, 2)^n$  with the cubes from the tiling  $[0, 2)^n + T$ . Let  $\mathcal{F}(n) = \mathcal{F} = \{[0, 2)^n \cap ([0, 2)^n + t) : t \in T\}$ . Since  $[0, 2)^n + T$  is a tiling,  $\mathcal{F}$  is a partition of the cube  $[0, 2)^n$  into boxes. Let  $m(K)$  denote the volume of the box  $K \in \mathcal{F}$ , and let  $m(\mathcal{F}) = \sum_{K \in \mathcal{F}} m(K)$ . For every  $K \in \mathcal{F}$  we have  $m(K) = 2^k$ , where  $k$  is the number of 1s in the vector  $v \in V$  such that  $K = [0, 2)^n \cap ([0, 2)^n + v - \mathbf{1})$ . Let  $M_k = |\{K \in \mathcal{F} : m(K) = 2^k\}|$ . The family  $\mathcal{F}$  is a partition of  $[0, 2)^n$  and therefore  $m(\mathcal{F}) = 2^n$

and

$$m(\mathcal{F}) - 2 = \sum_{k \geq 1} M_k 2^k.$$

Note now that if  $v \in V$  contains 3 at some position  $i \in [n]$ , then the cubes  $[0, 2]^n + v - \mathbf{1}$  and  $[0, 2]^n$  are disjoint. This means that these two cubes intersect if and only if  $v \in U \cup \{(0, \dots, 0), (2, \dots, 2)\}$ , where  $U$  consists of all sums of non-adjacent rows of the matrix  $A$  where the first and last rows are treated as adjacent. Thus,  $M_k$  is the number of all  $k$ -element subsets of the set  $\{1, \dots, n\}$  without two consecutive integers and which do not contain the pair 1,  $n$ . Hence,  $M_k = \binom{n-k}{k} \frac{n}{n-k}$ . This completes the proof of (1.1) for  $n \geq 3$  odd.  $\square$

This proofs needs only the portion  $U = U(n)$  of the Lagarias-Shor cube tiling code, where the code  $U$  is as in the previous proof: it consists of all sums of non-adjacent rows of the matrix  $A(n)$ , where the first and last rows are treated as adjacent. For  $n = 5$  we have

$$A(5) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad U(5) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

This is easily seen that making in every vector  $v \in U \cup \{(0, \dots, 0)\}$  the substitution  $2 \rightarrow 0$  we obtain the set of all vertices in the Lucas cube  $\Lambda_n$ .

To prove (1.2) and (1.3) for  $n \geq 3$  odd let  $\mathcal{F}_0^i, \mathcal{F}_2^i$  and  $\mathcal{F}_1^i, i \in [n]$  denote the sets of all boxes in  $\mathcal{F}$  which have the factors  $[0, 1), [1, 2)$  and  $[0, 2)$  at the  $i$ th position, respectively. Since  $\mathcal{F} = \{[0, 2]^n \cap ([0, 2]^n + v - \mathbf{1}) : v \in U \cup \{(0, \dots, 0), (2, \dots, 2)\}\}$ , for every  $k \in \{0, 1, 2\}$  the set  $\mathcal{F}_k^i$  consists of all boxes in  $\mathcal{F}$  which are determined by the vectors  $v \in U \cup \{(0, \dots, 0), (2, \dots, 2)\}$  such that  $v_i = k$ . Let

$$m(\mathcal{F}_{02}^i) = \sum_{K \in \mathcal{F}_{02}^i} m(K) \quad \text{and} \quad m(\mathcal{F}_1^i) = \sum_{K \in \mathcal{F}_1^i} m(K),$$

where  $\mathcal{F}_{02}^i = \mathcal{F}_0^i \cup \mathcal{F}_2^i$ .

The partition  $\mathcal{F}$  (Figure 1) has the structure which will be utilized below. Note that for every  $i \in [n]$  the set  $\bigcup \mathcal{F}_{02}^i$ , in an  $i$ -cylinder, i.e., for every line segment  $l_i \subset [0, 2]^n$  of length 2 which is parallel to the  $i$ th edge of the cube  $[0, 2]^n$  we have

$$l_i \subset \bigcup \mathcal{F}_{02}^i \text{ or } l_i \cap \bigcup \mathcal{F}_{02}^i = \emptyset. \quad (2.1)$$

Obviously, the set  $\bigcup \mathcal{F}_1^i$  is an  $i$ -cylinder too.

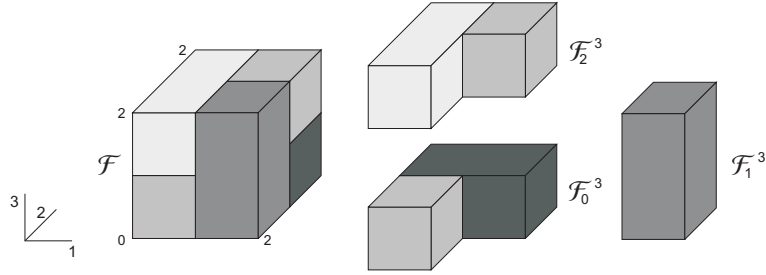


Fig. 1. The boxes in  $\mathcal{F}$  are determined by the vectors  $U = \{(1, 2, 0), (0, 1, 2), (2, 0, 1)\}$  (the three "long" boxes) and  $\{(0, 0, 0), (2, 2, 2)\}$  (the two unit cubes).

*Proof of (1.2) and (1.3) for  $n \geq 3$  odd.* We will calculate  $m(\mathcal{F}_{02}^i)$  and  $m(\mathcal{F}_1^i)$ . For every  $v \in U$  we have  $v_i = 1$  if and only if  $v_{i+1} = 2$ . Thus,  $\mathcal{F}_2^{i+1} = \mathcal{F}_1^i \cup \{[1, 2]^n\}$  and then  $m(\mathcal{F}_2^{i+1}) = m(\mathcal{F}_1^i) + 1$  (clearly,  $n + 1$  is taken modulo  $n$ ). It follows from (2.1) that  $m(\mathcal{F}_0^i) = m(\mathcal{F}_2^i)$  (see Figure 1). Since  $A$  is a circulant matrix, we have  $m(\mathcal{F}_1^i) = m(\mathcal{F}_1^j)$  for  $i, j \in [n]$ , and  $m(\mathcal{F}) = m(\mathcal{F}_0^i) + m(\mathcal{F}_2^i) + m(\mathcal{F}_1^i)$  because  $\mathcal{F}$  is a partition. Thus,  $2^n = 3m(\mathcal{F}_1^i) + 2$  and consequently

$$m(\mathcal{F}_{02}^i) = 2/3(2^n + 1) \quad \text{and} \quad m(\mathcal{F}_1^i) = 1/3(2^n - 2). \quad (2.2)$$

As it was noted before the proof, the code  $U \cup \{(0, \dots, 0)\}$  can be identify with the set of all vertices in the Lucas cube  $\Lambda_n$ .

Since  $\binom{n-k}{k} \frac{k}{n-k}$  is the number of all vertices of weight  $k$  in  $\Lambda_n$  with 1 at the first position, we have  $|\mathcal{F}_1^1| = \sum_{k \geq 1} \binom{n-k}{k} \frac{k}{n-k}$ , and consequently

$$m(\mathcal{F}_1^1) = \sum_{k \geq 1} \binom{n-k}{k} \frac{k}{n-k} 2^k,$$

which, by (2.2), gives (1.3) for  $n \geq 3$  odd. Since

$$\sum_{k \geq 0} \binom{n-k}{k} \frac{n}{n-k} 2^k = \sum_{k \geq 0} \binom{n-k}{k} 2^k + \sum_{k \geq 0} \binom{n-k}{k} \frac{k}{n-k} 2^k,$$

the proof of the identity (1.2) for  $n \geq 3$  odd is also completed.  $\square$

For  $n \geq 3$  odd all three identities are strongly related to the partition  $\mathcal{F}$ . The sums  $\sum_{k \geq 1} \binom{n-k}{k} 2^k + 2$  and  $\sum_{k \geq 1} \binom{n-k}{k} \frac{k}{n-k} 2^k$  are the total volumes of the boxes from the partition  $\mathcal{F}$  which belong to the sets  $\mathcal{F}_{02}^i$  and  $\mathcal{F}_1^i$ , respectively (the number 2 in the first sum is the sum of the volumes of the boxes  $[0, 1]^n$  and  $[1, 2]^n$ ). The summands  $\binom{n-k}{k} 2^k$  and  $\binom{n-k}{k} \frac{k}{n-k} 2^k$  for  $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$  are the total volumes of the boxes in  $\mathcal{F}_{02}^i$  and  $\mathcal{F}_1^i$ , respectively which have exactly  $k$  factors  $[0, 2)$ .

The identities (1.1)–(1.3) for  $n - 1 \geq 2$  even can be derived from the partitions  $\mathcal{F}(n)$  and  $\mathcal{F}(n - 2)$ , where  $\mathcal{F}(1) = \{[0, 1), [1, 2)\}$ .

*Proofs of (1.1)–(1.3) for  $n - 1 \geq 2$  even.* Denote by  $r_1, \dots, r_n$  the rows of the matrix  $A(n)$ , and let  $\mathcal{G} = \mathcal{G}(n - 1) \subset \mathcal{F}(n)$  be the set of all boxes which are determined by the vectors  $v \in U(n)$  which are sums of non-adjacent rows from the set  $\{r_1, \dots, r_{n-1}\}$ , where  $r_1$  and  $r_{n-1}$  are treated as adjacent. Thus, the number  $|\mathcal{G}|$  is the same as the number of all vertices in the Lucas cube  $\Lambda_{n-1}$ . Consequently

$$m(\mathcal{G}) = \sum_{k \geq 1} \binom{n-1-k}{k} \frac{n-1}{n-1-k} 2^k.$$

Since  $\mathcal{G}_0^n = \mathcal{F}_0^n \setminus \{[0, 1)^n\}$ , it follows that  $m(\mathcal{G}_0^n) = m(\mathcal{F}_0^n) - 1$ , and by (2.2) and the fact that  $m(\mathcal{F}_0^n) = m(\mathcal{F}_2^n)$  we get

$$m(\mathcal{G}_0^n) = 2/3(2^{n-1} - 1).$$

We now calculate  $m(\mathcal{G}_2^n)$ . Every box in  $\mathcal{G}_2^n$  is generated by a vector  $v \in U$  which has 2 at the  $n$ th position. Therefore,  $v = r_{n-1} + \sum_{i \in I} r_i$  for some  $I \subset \{2, \dots, n-3\}$ . Let  $R$  be the set of all such sums  $\sum_{i \in I} r_i$ . Every word in  $R$  is a sum of non-adjacent rows from the set  $\{r_2, \dots, r_{n-3}\}$ , where  $r_2$  and  $r_{n-3}$  are not treated as adjacent. Let  $U_0^{n-2}(n-2)$  be the set of all vectors in  $U(n-2)$  having 0 at the last position. Observe now that the function  $b : R \rightarrow U_0^{n-2}(n-2)$  defined by the formula  $b(u) = \sum_{i \in I-1} h_i$ , where  $h_1, \dots, h_{n-2}$  are rows of the matrix  $A(n-2)$  and  $I-1 = \{i-1 : i \in I\}$ , is a bijection. Therefore,  $m(\mathcal{G}_2^n) = 2m(\mathcal{F}_0^{n-2}(n-2))$  (recall that we add  $r_{n-1}$  to  $\sum_{i \in I} r_i$ ). By (2.1),  $m(\mathcal{F}_0^{n-2}(n-2)) = m(\mathcal{F}_2^{n-2}(n-2))$ , and by (2.2),

$$m(\mathcal{G}_2^n) = 1/3(2^{n-1} + 2).$$

Thus,  $m(\mathcal{G}) = m(\mathcal{G}_0^n) + m(\mathcal{G}_2^n) = 2^{n-1}$  because  $\mathcal{G}_1^n = \emptyset$ . This completes the proof of (1.1) for  $n - 1 \geq 2$  even.

Since  $m(\mathcal{G}_2^{i+1}) = m(\mathcal{G}_1^i)$ ,  $m(\mathcal{G}_1^i) = m(\mathcal{G}_1^j)$  and  $m(\mathcal{G}) = m(\mathcal{G}_1^i) + m(\mathcal{G}_{02}^i)$  for  $i, j \in [n-1]$ , it follows that

$$m(\mathcal{G}_1^i) = 1/3(2^{n-1} + 2) \quad \text{and} \quad m(\mathcal{G}_{02}^i) = 2/3(2^{n-1} - 1)$$

for  $i \in [n-1]$ .

By the definition of the set  $\mathcal{G}$ , we have  $|\mathcal{G}_1^1| = \sum_{k \geq 1} \binom{n-1-k}{k} \frac{k}{n-1-k}$ , and thus

$$m(\mathcal{G}_1^1) = \sum_{k \geq 1} \binom{n-1-k}{k} \frac{k}{n-1-k} 2^k$$

which proves (1.3) for  $n-1 \geq 2$  even. Having this, in the same manner as for  $n \geq 3$  odd we prove (1.2) for  $n-1 \geq 2$  even.  $\square$

**REMARK 2.1** There are many tiling proofs that rely on counting the number of 1-dimensional tilings of a  $1 \times n$  board by polyominoes (squares, dominoes etc.) (see [1, 2]). In our case we examine just one  $n$ -dimensional tiling of  $\mathbb{R}^n$  by translates of the cube  $[0, 2)^n$ , and especially it is exploited the structure of that tiling.

At the end we show that for  $n \geq 3$  odd the set of all vertices of the Lucas cube  $\Lambda_n$  is a selector of a discrete analogue of the partition  $\mathcal{F}$

### 3 Vertices of the Lucas cube as a selector

Let  $L \cup \{(0, \dots, 0), (1, \dots, 1)\}$  be the code that arises from  $U$  by making in every vector  $v \in U \cup \{(0, \dots, 0), (2, \dots, 2)\}$  the following substitutions:  $0 \rightarrow 0$ ,  $2 \rightarrow 1$  and  $1 \rightarrow *$  For example, for  $n = 5$  we have

$$L = \begin{bmatrix} * & 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 \\ 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & * & 1 \\ 1 & 0 & 0 & 0 & * \\ * & 1 & * & 1 & 0 \\ * & 1 & 0 & * & 1 \\ 0 & * & 1 & * & 1 \\ 1 & * & 1 & 0 & * \\ 1 & 0 & * & 1 & * \end{bmatrix}.$$

The set  $L$  consists of all sums of non-adjacent rows of the matrix  $\text{circ}(*, 1, 0, \dots, 0)$ , where the first and last rows are treated as adjacent. Therefore, making the substitution  $* \rightarrow 0$  in every vector of  $L \cup \{(0, \dots, 0)\}$  we obtain the set  $V(\Lambda_n)$  of all vertices in the Lucas cube  $\Lambda_n$ .

The code  $L \cup \{(0, \dots, 0), (1, \dots, 1)\}$  induces a partition  $\mathcal{L}$  of discrete box  $\{0, 1\}^n$  into boxes which is a discrete analogue of the partition  $\mathcal{F}$  from the previous section. The boxes  $K(l) = K_1(l) \times \dots \times K_n(l) \in \mathcal{L}$ , where  $l \in L \cup \{(0, \dots, 0), (1, \dots, 1)\}$ , are of the form:

$$K_i(l) = \begin{cases} \{0\} & \text{if } l_i = 0, \\ \{1\} & \text{if } l_i = 1, \\ \{0, 1\} & \text{if } l_i = * \end{cases}$$

for  $i \in [n]$ .

Observe now that for every  $n \geq 3$  odd the set  $V(\Lambda_n)$  of the vertices of the Lucas cube is a *selector* of the family of boxes  $\mathcal{L} \setminus \{\{1\} \times \dots \times \{1\}\}$ : for every  $v \in V(\Lambda_n)$  there is exactly one  $K(l) \in \mathcal{L} \setminus \{\{1\} \times \dots \times \{1\}\}$  such that

$$v \in K(l).$$

Indeed, let  $K(l) \in \mathcal{L} \setminus \{\{1\} \times \dots \times \{1\}\}$  and pick a point  $v = v_i \dots v_d \in K(l)$  in the following way:

$$v_i = \begin{cases} 0 & \text{if } K_i(l) = \{0\}, \\ 1 & \text{if } K_i(l) = \{1\}, \\ 0 & \text{if } K_i(l) = \{0, 1\}. \end{cases}$$

Since  $l$  does not contain two consecutive 1s modulo  $n$  and if  $K_i = \{0, 1\}$ , then  $K_{i+1} = \{1\}$ , it follows that  $v \in V(\Lambda_n)$  and for every  $w \in K(l)$ ,  $w \neq v$ , there is  $i \in [n]$  such that  $w_i = 1$  while  $v_i = 0$ . Thus,  $K(l) \cap V(\Lambda_n) = \{v\}$ .

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